Univalent Foundations
and the equivalence principle

Benedikt Ahrens

Outline

1. The equivalence principle

2. Invariance in Univalent Foundations
   - Overview of Univalent Foundations
   - Univalence Axiom: invariance under equivalence of types
   - Lifting invariance to groups
   - Categories and equivalence
   - Structure Identity Principle

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Indiscernability of identicals

Identical objects satisfy the same properties

$$x = y \;\; \rightarrow \;\; \forall P \, (P(x) \leftrightarrow P(y))$$

- Reasoning **in logic** is invariant under equality
- **In mathematics**, reasoning should be invariant under weaker notion of sameness!

The equivalence principle

Equivalence principle

Reasoning in mathematics should be **invariant under** the appropriate notion of **sameness**.

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Equivalence principle

Reasoning in mathematics should be **invariant under** the appropriate notion of **sameness**.

Notion of sameness depends on the objects under consideration:

- **equal** numbers, functions,...
- **isomorphic** sets, groups, rings,...
- **equivalent** categories
- **biequivalent** bicategories
- ...

Non-examples: statements violating equivalence principle

We can easily **violate** this principle:

Exercise

Find a statement about categories that is not invariant under the equivalence of categories

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A solution

"The category $\mathcal{C}$ has exactly one object."

Maybe this statement is simply silly!

A language for invariant properties

M. Makkai, *Towards a Categorical Foundation of Mathematics*:

> *The basic character of the Principle of Isomorphism is that of a **constraint on the language** of Abstract Mathematics; a welcome one, since it provides for the separation of sense from nonsense.*

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Goal

to have a **syntactic criterion** for properties and constructions that are invariant under equivalence

How to break the invariance principle for categories…

- Recall: the statement

  The category $\mathcal{C}$ has exactly one object.

  is not invariant under equivalence of categories.
- In general, referring to **equality of objects** breaks invariance, but…

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- even the **definition** of category refers to equality of objects:

Problem

"If $\mathsf{source}(g)$ is **equal to** $\mathsf{target}(f)$, then $g \circ f$ exists."

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"If $\mathsf{source}(g)$ is **equal to** $\mathsf{target}(f)$, then $g \circ f$ exists."

Can we give a definition of category without equality of objects?

…and how to fix it.

Solution

Use a logic/language of **dependent types**, in which $s(g) = t(f)$ is encoded by what type of thing f and g are.

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A category consists of

- a collection O of objects
- for each $x, y \in O$, a collection $A(x, y)$ of arrows
- for each $x, y, z \in O$ and each $f \in A(x, y)$ and $g \in A(y, z)$, a composite $g \circ f \in A(x, z)$
- for each $x \in O$, an identity $\mathsf{id}_x \in A(x, x)$
- …

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Gives rise to **dependently typed language** by adding logical connectors.

Invariance for properties

Theorem (Freyd '76, Blanc '78)

A property of categories (expressed in 2-sorted first order logic) is invariant under equivalence iff it can be expressed in this dependently typed language, using equality for arrows but not for objects.

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- What about **constructions** on categories?

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- What about **constructions** on categories?
- What about other mathematical structures?

Equivalence principle in the univalent foundations

In the univalent foundations

- an equivalence principle can be proved for a variety of structures
  - sets
  - groups, rings, . . .
  - categories
- EP applies not only to properties, but also to constructions: any construction transports suitably along equivalence

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Overview: types in univalent type theory

Type former	Notation	(special case)
Inhabitant	$a : A$	
Dependent type	$x : A \;\vdash\; B(x)$	
Sigma type	$\sum_{(x:A)} B(x)$	$A \times B$
Product type	$\prod_{(x:A)} B(x)$	$A \to B$
Coproduct type	$A + B$	
Identity type	$\mathsf{Id}_A(a, b)$, $a = b$	
Universe	U	
Base types	Nat, Bool, $\mathbf{1}$, $\mathbf{0}$	

and some axioms: function extensionality, univalence

Transport

For a given dependent type

$$x : A \ \vdash \ B(x)$$

and $a, b : A$, the rules of the identity type allow to construct a term

$$\mathsf{transport}^B : B(a) \times (a = b) \to B(b)$$

Contractible types, propositions and sets

- A is **contractible** if we can construct a term of type

$$\mathsf{isContr}(A) \stackrel{\text{def}}{=} \sum_{(x:A)} \prod_{(y:A)} y = x$$

- A is a **proposition** if

$$\mathsf{isProp}(A) \stackrel{\text{def}}{=} \prod_{x,y:A} \mathsf{isContr}(x = y)$$

- A is a **set** if

$$\mathsf{isSet}(A) \stackrel{\text{def}}{=} \prod_{x,y:A} \mathsf{isProp}(x = y)$$

$$\mathsf{Prop} \stackrel{\text{def}}{=} \sum_{x:\mathsf{U}} \mathsf{isProp}(X) \qquad \mathsf{Set} \stackrel{\text{def}}{=} \sum_{X:\mathsf{U}} \mathsf{isSet}(X)$$

Equivalences

Definition

A map $f : A \to B$ is an **equivalence** if it has contractible fibers, i.e.,

$$\mathsf{isequiv}(f) \stackrel{\text{def}}{=} \prod_{b:B} \mathsf{isContr}\left(\sum_{a:A} f(a) = b\right)$$

The type of equivalences:

$$A \simeq B \stackrel{\text{def}}{=} \sum_{f:A\to B} \mathsf{isequiv}(f)$$

Characterizing some identity types

Can construct equivalences

- for $f, g : A \to B$

$$\Big(f = g\Big) \simeq \Big(\prod_{a:A} f(a) = g(a)\Big)$$

- for $s, t : A \times B$

$$\Big(s = t\Big) \simeq \Big(\big(\mathsf{pr}_1(s) = \mathsf{pr}_1(t)\big) \times \big(\mathsf{pr}_2(s) = \mathsf{pr}_2(t)\big)\Big)$$

- for $s, t : \sum_{(x:A)} B(x)$

$$\Big(s = t\Big) \simeq \left(\sum_{e:\mathsf{pr}_1(s)=\mathsf{pr}_1(t)} \mathsf{transport}^B\Big(e,\ \mathsf{pr}_2(s)\Big) = \mathsf{pr}_2(t)\right)$$

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Universes

There is a type U that contains all types, i.e., $A : \mathsf{U}$.

- Actually, hierarchy $(\mathsf{U}_i)_{i \in I}$ to avoid paradoxes.
- a dependent type

$$x : A \ \vdash \ B : \mathsf{U}$$

can be considered as a function

$$\lambda x.B : A \to \mathsf{U}$$

Question

What is

$$\mathsf{Id}_{\mathsf{U}}(A, B) \quad ?$$

Voevodsky's Univalence Axiom

Naïve answer

$$\mathsf{univalence} : (A =_{\mathsf{U}} B) \simeq (A \simeq B)$$

More controlled:

Answer

Define

$$\mathsf{idtoeqv} : \prod_{A,B:\mathsf{U}} (A = B) \to (A \simeq B)$$

$$\mathsf{refl}_A \mapsto \mathsf{id}$$

$$\text{Axiom } \mathsf{univalence} : \prod_{A,B:\mathsf{U}} \mathsf{isequiv}(\mathsf{idtoeqv}_{A,B})$$

Invariance under equivalence

For a given predicate $P : \mathsf{U} \to \mathsf{U}$ and $A, B : \mathsf{U}$, from

$$\mathsf{transport}^P : P(A) \times (A = B) \to P(B)$$

and

$$(A =_{\mathsf{U}} B) \simeq (A \simeq B)$$

obtain

$$P(A) \times (A \simeq B) \to P(B)$$

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Groups in Univalent Foundations

A **group** $G = (X, S)$ in Univalent Foundations is

- a set X
- operations

$$\begin{array}{ccccc} & & X \times X & & \\ & & \downarrow{\scriptstyle m} & & \\ X & \xrightarrow{i} & X & \xleftarrow{e} & 1 \end{array}$$

- such that group axioms are satisfied

The type of groups is

$$\mathsf{Grp} := \sum_{X : \mathsf{Set}} \mathsf{GrpStructure}(X)$$

Lifting univalence from types to groups

A **group isomorphism** $G \to G'$ is

- a bijective function on the underlying types $X \to X'$
- compatible with the group structures S and S' on X and X'.

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Theorem (EP on types lifts to EP on groups)

- *An isomorphism of groups lifts to an equivalence of all constructions on groups (in UF):*

$$\prod_{(P:\mathsf{Grp}\to\mathsf{U})} \prod_{(G,G':\mathsf{Grp})} (G \cong G') \times P(G) \to P(G')$$

- *In particular: any statement about groups is invariant under group isomorphism*

Lifting univalence from types to groups

The proof of this statement uses 2 ingredients:

1. $(G = G') \;\simeq\; (G \cong G')$
2. Transport along identities

$(G = G') \;\simeq\; (G \cong G')$ is given by the canonical map

$$\mathsf{refl}_G \mapsto \mathsf{id}_G$$

Identity is isomorphism for groups

$$
\begin{aligned}
G = G' &\simeq (X, S) = (X', S') \\
&\simeq \sum_{p : X = X'} \mathsf{transport}^{\mathsf{GrpStructure}}(p, S) = S' \\
&\simeq \sum_{p : X = X'} (\mathsf{transport}^{Y \mapsto (Y \times Y \to Y)}(p, m) = m') \\
&\qquad\qquad \times (\mathsf{transport}^{Y \mapsto (Y \to Y)}(p, i) = i') \\
&\qquad\qquad \times (\mathsf{transport}^{Y \mapsto (1 \to Y)}(p, e) = e') \\
&\simeq \sum_{f : X \simeq X'} \left(f \circ m \circ (f^{-1} \times f^{-1}) = m'\right) \\
&\qquad\qquad \times \left(f \circ i \circ f^{-1} = i'\right) \\
&\qquad\qquad \times (f \circ e = e') \\
&\simeq (G \cong G')
\end{aligned}
$$

Lifting univalence to algebraic structures

Lifting univalence to algebraic structures (Aczel, Coquand, Danielsson)

For many algebraic structures in univalent foundations, univalence lifts.

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For many algebraic structures in univalent foundations, univalence lifts.

Examples include:

- rings
- posets
- discrete fields
- sets with fixpoint operator

This general result is best explained in terms of categories; let's have a look at those first.

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Univalence for categories

Univalence for algebraic structures

For groups, rings, etc., univalence lifts.

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Going to equivalence of categories:

Univalence for categories

For **univalent** categories, equivalence is identity via canonical map

$$(\mathcal{C} = \mathcal{D}) \;\simeq\; \mathsf{Equiv}(\mathcal{C}, \mathcal{D}) \;.$$

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That is, any construction on univalent categories in Univalent Foundations is invariant under **equivalence**.

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For **univalent** categories, equivalence is identity via canonical map

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That is, any construction on univalent categories in Univalent Foundations is invariant under **equivalence**.

- What is a category in UF?
- What is the **univalence** condition for categories?

Categories in univalent type theory

A category is

- a type $O : \mathsf{U}$ of objects
- a dependent type $A : O \times O \to \mathcal{S}et$ of arrows
- $\mathsf{id} : \prod_{(a:O)} A(a,a)$
- $(\circ) : \prod_{(a,b,c:O)} A(a,b) \times A(b,c) \to A(a,c)$
- axioms postulating identities of arrows

Categories in univalent type theory

A **univalent** category is

- a type $O : \mathsf{U}$ of objects
- a dependent type $A : O \times O \to \mathcal{S}et$ of arrows
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- $(\circ) : \prod_{(a,b,c:O)} A(a,b) \times A(b,c) \to A(a,c)$
- axioms postulating identities of arrows
- such that the natural map

$$\mathsf{idtoiso} : \prod_{a,b:O} (a = b) \to \mathsf{iso}(a,b)$$

is an equivalence for any a, b

Some remarks on univalent categories

In simplicial set model

- categories correspond to truncated Segal spaces (Rezk, A model for the homotopy theory of homotopy theory)
- univalence corresponds to completeness

Connection with Freyd, Blanc

Instead of avoiding identity of objects as in [F,B], in a univalent category, identity means (can be replaced with) isomorphism.

Examples of univalent categories

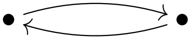
- $\mathcal{S}et$ (discrete types)
- Groups, rings, … (Structure Identity Principle)
- Functor category $[\mathcal{C}, \mathcal{D}]$, if $\mathcal{D}$ is univalent
- Full subcategories of univalent categories

More examples of univalent categories

- A preorder is univalent iff it is antisymmetric
- If X is of h-level 3, i.e., 1-truncated, then there is a univalent category with X as objects and $\hom(x, y) := (x = y)$
- If $\mathcal{C}$ is univalent, then the category of cones of shape $F : \mathcal{J} \to \mathcal{C}$ is
  - $\leadsto$ limits (limiting cones) in a univalent category are unique **up to identity**

Non-univalent categories

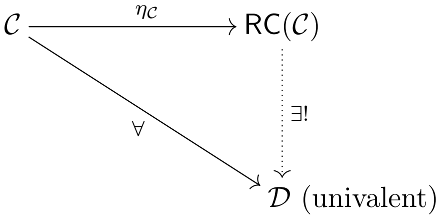
- Any "chaotic" category $\mathcal{C}$ with $\mathcal{C}(x,y) := 1$, for $\mathcal{C}_0$ not a prop



- Any chaotic category $\mathcal{C}$ with an object $c : \mathcal{C}_0$ is **equivalent** to the terminal category $\mathbf{1}$
  - $\rightsquigarrow$ a category can be equivalent to a univalent one without being univalent itself

Rezk completion

To any category $\mathcal{C}$, associate a univalent one, its "Rezk completion",



Intuitively, obtain $\mathsf{RC}(\mathcal{C})_0$ by adding to $\mathcal{C}_0$ as many identities as needed

Construction of the Rezk completion

- $\mathsf{RC}(\mathcal{C})$ is the full image subcategory of the Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{S}et]$
- $\eta_{\mathcal{C}} : \mathcal{C} \to \mathsf{RC}(\mathcal{C})$ is fully faithful and essentially surjective
- precomposition with a ff. and es. functor is ff. and es.
- a ff. and es. functor is an equivalence if source category is univalent
- the object map of an equivalence of univalent categories is an equivalence of types

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Structure identity principle in detail

Reminder: Key for the equivalence principle for groups is the equivalence

$$(G = G') \;\simeq\; (G \cong G')$$

given by

$$\mathsf{refl}_G \mapsto \mathsf{id}_G$$

This is nothing else than saying that the category of groups is univalent.

Groups as a structure on sets

- A group is a set with some structure
- A group morphism is a map of sets compatible with that structure

Univalence of the category of groups comes from

- univalence of category of sets
- the extra structure on sets and maps that defines groups and group homomorphisms is “good”

Structures on a category

Let $\mathcal{C}$ be a category. Call (P, H)-structure

- a predicate $P : \mathcal{C}_0 \to \mathsf{U}$ on objects of $\mathcal{C}$
- for any $x, y : \mathcal{C}_0$ and $a : P(x)$ and $b : P(y)$ and $f : \mathcal{C}(x, y)$, a proposition

$$H_{a,b}(f) : \mathsf{Prop}$$

- for any $x : \mathcal{C}_0$ and $a : P(x)$, have

$$H_{a,a}(1_x)$$

- have

$$H_{a,b}(f) \to H_{b,c}(g) \to H_{a,c}(g \circ f)$$

- plus another condition on H

Structure Identity Principle

Theorem

If $\mathcal{C}$ is univalent, and (P, H) is a structure as before, then the following category is univalent:

objects *pairs of an object $x : \mathcal{C}_0$ and a P-structure on x:*

$$(x, a) : \sum_{x:\mathcal{C}} P(x)$$

morphisms *morphisms from (x, a) to (y, b) are $f : \mathcal{C}(x, y)$ that satisfy H,*

$$(f, p) : \sum_{f:\mathcal{C}(x,y)} H_{a,b}(f)$$

Examples of “good” structures

The “group” (P, H) structure

- $P(X) :=$ group structures on X
- $H_{a,b}(f) :=$ “f is compatible with group structures a and b”

Analogously for many other algebraic structures ...

Summary

- univalence axiom asserts equivalence principle (EP) for types

- EP for types lifts to EP for groups etc.

- EP for higher-categorical structures requires an additional restriction (e.g., EP for **univalent** categories)

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Thanks for your attention!

Some references

- Freyd, *Properties invariant within equivalence types of categories*
- Blanc, *Equivalence naturelle et formules logiques en théorie des catégories*
- Coquand, Danielsson, *Isomorphism is equality*
- Kapulkin, Lumsdaine, *The Simplicial Model of Univalent Foundations (after Voevodsky)*
- Rezk, *A model for the homotopy theory of homotopy theory*
- The HoTT book, `https://homotopytypetheory.org/book/`
- Ahrens, Kapulkin, Shulman, *Univalent categories and the Rezk completion* `http://arxiv.org/abs/1303.0584`