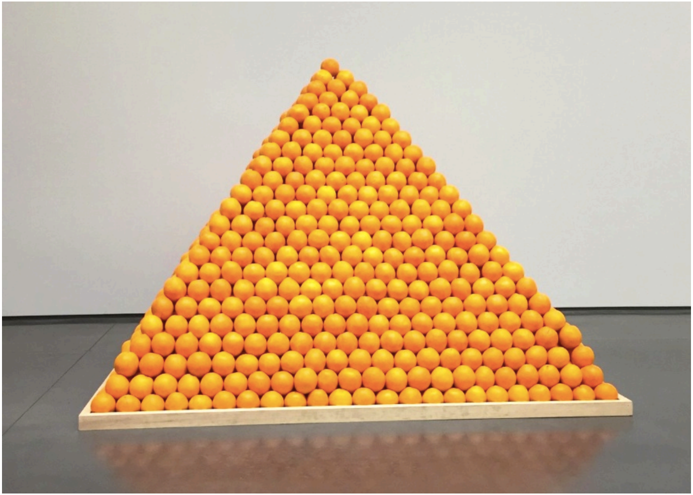
Proving Theorems from Reflection: Global Reflection Principles



P.D. Welch, University of Bristol, FOMUS Conference, 22.vi.16, Bielefeld

- Part I: Description of some problems on projective sets.
- Part II: Large cardinals or strong axioms of infinity.
- Part III: Global Reflection Principles on the universe of sets.

Part I: Some interesting problems on *projective sets*

Definition (Borel Sets)

Let T be a Polish space; let B_0 be the class of closed sets in T ;
Let $B_\eta = \{\bigcup_{n\in\mathbb{N}} A_n \mid \neg A_n \text{ in some } B_{\eta_n} \text{ for an } \eta_n < \eta\}$.
Let $\mathcal{B} = \bigcup_{\eta<\omega_1} B_\eta$.

Definition (Analytic Sets)

Let T be a Polish space; let B_0 be the class of closed sets in T ;
Let $\mathcal{A} =_{df} \{A \mid \exists C \in B_0 \in T \times T(A = proj(C)\}$

where $proj(C) = \{x \mid \exists y \in T : C(x, y)\}$.

Theorem (Suslin)

Borel $= \mathcal{A} \cap co\text{-}\mathcal{A}$

Definition (Luzin: Projective Sets)

$S_1 = \mathcal{A}$ (in any dimension);
$S_{n+1} = \{proj(D) \mid D \subseteq T^k \times T, D \in co\text{-}S_n\}$
$PROJ = \bigcup_n S_n$.

Theorem (Suslin)

Any $D \in \mathcal{A}$ is Lebesgue measurable.

Problem (1 Lebesgue Measurability)

Are the sets in *PROJ* Lebesgue measurable?

Quote: (Luzin - 1925) “One does not know and one will never know whether the projective sets are *LM*”.

The Baire and Perfect Subset properties

Problem (2 Property of Baire)

Do sets in *PROJ* have the property of Baire (BP)?

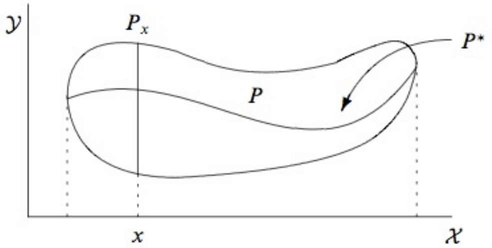
(BP): has meagre symmetric difference with some open set.

Problem (3 Perfect subset property (PSP))

Does every uncountable set in *PROJ* contain a perfect set?

(Since a perfect set has size the continuum, Cantor's continuum problem is settled for such sets.)

Uniformisation



A function $P^* \subseteq P$ *uniformizes* P if

$$\forall x[\exists y(x,y) \in P \rightarrow \exists\, y'\, (P^*(x) = y')].$$

A function P^* is *projective* if its graph is.

Problem (4 Uniformisation Property (*Unif*))

Does every set P in $T \times T$ in *PROJ* have a projective uniformiser? $\mathit{Unif}(\mathit{PROJ})$?

Theorem (Novikov-Kondō 1937)

Every co-analytic subset of the plane has a co-analytic uniformiser.

Banach-Tarski

The above properties of the projective sets are called the *regularity properties*.

Problem (5 Banach -Tarski Problem)

Is there a paradoxical decomposition of the sphere in R^n into projective pieces?

To summarise:

P1: $LM(PROJ)$
P2: $BP(PROJ)$
P3: $PSP(PROJ)$
P4: $Unif(PROJ)$
P5: Banach Tarski with projective pieces

Regularity Properties can consistently fail:

Theorem

(Gödel) If *ZF* is consistent, then so is
ZFC+*"There is a projective set that is not LM"*.

• Indeed there is a *projection of a co-analytic* ("*PCA*") set that fails to be *LM*. This gives a negative "answer" to P1.

and also to:

P2: In Gödel's constructible universe there is a *PCA*-set without the Baire property BP.

P5: In Gödel's constructible universe there is a paradoxical decomposition of the unit sphere in $\mathbb{R}^3$ using *PCA*-pieces.

For P3:

Theorem

(Gödel) If *ZF* is consistent, then so is *ZFC+“There is a co-analytic set that is uncountable with no perfect subset”*.

A different story

• Are there principles that give a fuller picture of the regularity properties? By Gödel's results, these must be principles that go beyond *ZFC*.

• This can be done using an assumption on the *determinacy of infinite games*.

Part II: An early clue

Theorem (Solovay)

ZF proves that if there is a $<$-κ-additive 2-valued measure on some set of cardinality $\kappa > \aleph_0$ then $BP(PCA)$, $LM(PCA)$, $PSP(PCA)$.

- These are then in contradiction to the picture given in Gödel's L.

Determinacy of Gale-Stewart games

Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ (or some $X^{\mathbb{N}}$). The game G_A is defined as follows:

$$\begin{array}{ll} I \text{ plays } k_0 \quad k_2 \quad \dots k_{2n} \dots \\ II \text{ plays } \quad k_1 \quad k_3 \quad \dots k_{2n+1} \dots \end{array}$$

- *I wins* if and only if $x = (k_0, k_1, \dots) \in A$.
- G_A is *determined* if either Player has a winning strategy in this game.
- *Projective Determinacy (Det(PROJ))*

  " All G_A are determined for projective A".

Theorem (Mycielski-Steinhaus,-Swierckowski, Moschovakis)

$Det(PROJ)$ implies Regularity for the projective sets.

Theorem (Martin)

ZF proves that if there is a $<$-κ-additive 2-valued measure on some set of cardinality $\kappa > \aleph_0$ then $Det(Analytic)$.

This was much earlier than:

Theorem (Martin)

ZF proves $Det(Borel)$.

• But *ZFC* alone is just not strong enough to prove $Det(Analytic)$ on its own: this is because $Det(Analytic)$ can prove the consistency of *ZFC*. (And we cannot contradict Gödel's Incompleteness Theorems.)

After much effort the prize was won:

Theorem (Martin-Steel)

If there are infinitely many "Woodin" cardinal numbers then Det$(PROJ)$ and hence Regularity for the projective sets.

Problem: But how can we justify these cardinals?

Part III: Reflection Principles in Set Theory

To say that the universe of all sets is an unfinished totality does not mean objective undeterminateness, but merely a subjective inability to finish it.

Gödel, in (Wang: "A Logical Journey: From Gödel to Philosophy").

• Historically *reflection principles* are associated with attempts to say that no one notion, idea, statement can capture our whole view of $V = \bigcup_{\alpha \in On} V_\alpha$.

• Such reflection principles are usually formulated in some language (first or higher order) as sentences φ (when interpreted in the appropriate way over V) that hold in $\langle V, \in, \ldots \rangle$, hold in some $\langle V_\beta, \in, \ldots \rangle$ - *sentential* reflection.

• We suggest a *Global Reflection Principle* to overcome the limitations that these principles are all *intra-constructible*.

P. Koellner.
On reflection principles.
Ann. Pure Appl. Logic, 157:206–219, 2009.

D. A. Martin.
Completeness or Incompleteness of Basic Mathematical Concepts.
EFI Harvard Workshop Papers, 2012.

P.D. Welch.
Global Reflection Principles.
Isaac Newton Institute Pre-print Series., Exploring the Frontiers of Incompleteness, EFI Workshop Papers, Harvard,(INI12051-SAS), 2012.

P.D. Welch and L. Horsten.
Reflecting on Absolute Infinity.
Journal of Philosophy, to appear 2016.

The Challenge

- To rise to the challenge to justify a set-theoretic *reflection principle* that will ensure the existence of large cardinals (or strong axioms of infinity) that are sufficient to deliver the hypotheses needed for modern set theoretical principles.

Montague-Levy: 1st order Reflection

(R_0) : *For any* $\varphi(v_0, \ldots, v_n) \in \mathcal{L}_{\dot{\in}}$

$$ZF \vdash \forall\alpha\exists\beta > \alpha\forall\vec{x} \in V_\beta[\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x})^{V_\beta}].$$

Indeed by formalising a Σ_n-Satisfaction predicate we have:

For each n
ZF $\vdash \exists C_n[C_n \subseteq$ *On is a c.u.b. class so that for any* $\varphi \in \mathrm{Fml}_{\Sigma_n}$:

$$\forall\beta \in C_n\forall\vec{x} \in V_\beta[\varphi(\vec{x}) \leftrightarrow \varphi(\vec{x})^{V_\beta}]].$$

Informally we write $\forall\beta \in C_n : (V_\beta, \in) \prec_{\Sigma_n} (V, \in)$.

Levy, Bernays

Suppose we allow some *second order* methods and consider *classes*. If we allow reflection on classes then we can deliver some modest large cardinals. Let $\Phi(D)$ be the assertion that

"D is a function from On to On, but $\forall\alpha\, D``\alpha$ *is bounded in On"*.

By the Axiom of Replacement for any class D, we have: Then

$$(V, \in, D) \models \Phi(D).$$

If we allow the assumption that Φ *reflects* to some V_κ we shall have:

$$\forall D \subseteq V_\kappa\, (V_\kappa, \in, D) \models \Phi(D).$$

This implies that κ is an *inaccessible cardinal*.

• The strict *ZFC*-ist will eschew such an argument as it quantifies over classes that are not necessarily definable over $(V, \in)$.

Gödel again:

> *All the principles for setting up the axioms of set theory should be reducible to Ackermann's principle: The Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now."* (*Wang* - ibid.).

Strengthening Reflection Principles

• Strengthened Reflection principles (Tait): higher order (but these cannot have 3rd order parameters (Reinhardt)).

• The Reflection Principles to date are all consistent with a view of the universe as being L the constructible one: they are *intra-constructible.* (And Koellner, *op.cit.*, showed this for the strengthened principles of Tait.)

However these are all motivated on a syntactic level.

Moral: We need stronger Reflection Principles: those that generalise Montague-Levy are not up to the task of providing any justification for the large cardinals needed for modern set theory.

Why ask for stronger reflection principles?

Theorem (Woodin)

Suppose there is a proper class of Woodin cardinals. Then $Th(L(\mathbb{R}))$ is immune to change by set forcing.

• This supposition is now ubiquitous.

• We shall therefore define such a *Global Reflection Principle* (GRP) which will deliver such large cardinals and so $Det(PROJ)$.

• We take an almost naive Cantorian stance, and consider the *absolute infinities* that he identified at that time: the absolute infinity of *On* the ordinals, *Card* the class of cardinals, *V* itself ? *etc.*

• We collect these into a family $\mathcal{C}$.

• We consider reflection of the whole universe $(V, \in, \mathcal{C})$ to a small structure.

Global Reflection Principle - GRP

- We take a small (meaning *set-sized*) substructure of $(V, \in, \mathcal{C})$, the universe with all of its parts, $\mathcal{C}$, and ask that this is then isomorphic to a small part of V: namely some V_α together with all of its parts. The 'parts' of V_α are naturally those $D \subseteq V_\alpha$, that is $V_{\alpha+1}$.

Definition (Global Reflection Principle - GRP)

There is a set $X \subseteq V$ and a set-sized collection $\mathcal{C}' \subseteq \mathcal{C}$ with :

$$(X, \in, \mathcal{C}') \prec (V, \in, \mathcal{C})$$

and:

$$(X, \in, \mathcal{C}') \cong (V_\alpha, \in, V_{\alpha+1})$$

for some $\alpha \in On$. Hence

$$(V, \in, \mathcal{C}) \text{ is reflected down to } (V_\alpha, \in, V_{\alpha+1})$$

- This implies that: $V_{\alpha+1} = \{D \cap V_\alpha \mid D \in \mathcal{C}'\}$.)

Global Reflection Principle - GRP

We are thus requiring that there is set-sized simulacrum of

$(V, \in, \mathcal{C})$ *that is of the form* $(V_\alpha, \in, V_{\alpha+1})$.

Why (GRP)?

Define a field of classes U on $\mathcal{P}(\kappa)$ by

$$X \in U \leftrightarrow \kappa \in j(X)$$

where j is the isomorphism to the substructure $(X, \in, \mathcal{C}')$ of $(V, \in, \mathcal{C})$. As $\mathcal{P}(\kappa) \subseteq V_{\kappa+1} \subseteq \mathrm{dom}(j)$ by Σ_1-elementarity (in j), this is an ultrafilter.

• Standard arguments show that U is a normal measure on κ, and thus κ is a measurable cardinal.

But then:

$\forall\alpha < \kappa\ \langle V, \in\rangle \models$"$\exists\kappa > \alpha(\kappa$ a measurable cardinal)" $\Longrightarrow$
$\Longrightarrow \langle V_\kappa, \in\rangle \models$" $\forall\alpha\exists\lambda > \alpha(\lambda$ a measurable cardinal)" $\Longrightarrow$
$\Longrightarrow \langle V, \in\rangle \models$"*There are unboundedly many measurable cardinals*".

Consequences of the Global Reflection Principle

Theorem (GRP)

$(V, \in) \models \forall\alpha\exists\lambda > \alpha(\lambda$ *a measurable Woodin cardinal).*

- By results of Martin-Steel and Woodin, *GRP* then implies:
  - a) Projective Determinacy $Det(PROJ)$ and $(AD)^{L(\mathbb{R})}$.
  - b) (Woodin) $\mathrm{Th}(L(\mathbb{R}))$ is fixed: no set forcing notion can change $\mathrm{Th}(L(\mathbb{R}))$, and in particular the truth value of any sentence about reals in the language of analysis, thereby including $Det(PROJ)$.

Conclusion:

We can get a good mathematical *theory of the projective sets by allowing ourselves to take a* foundational theory of classes *seriously, and formulating reflection on a universe with those classes.*